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ABSTRACT

Recent reforms in mathematics education suggest that classroom teachers ought to shift away from the traditional concentration on arithmetical computation and adopt goals of mathematical confidence, problem solving, communication, reasoning, and appreciation. To understand how students are thinking in a given situation, teachers need to make their students' thinking public. One way of accomplishing this is to encourage students to engage in mathematical conversations. This report describes a fourth-grade classroom where students' thinking was made public through discussions in which students presented and justified their interpretations of, and solutions to, the problems presented in class. In the beginning, the author identified the goals the teacher had for her students and then followed the development of those skills and abilities through a series of classroom observations and teacher and student interviews. The teacher and her students learned to talk about mathematics in ways that made their thinking visible and suggested that they know mathematics in fresh, inventive ways. The report ends with questions about what students learned and what teachers and researchers might consider evidence of their learning. (Contains 22 references.) (Author)

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MAKING CONNECTIONS: TALKING AND LEARNING IN A FOURTH-GRADE CLASS

James W. Reineke

Center for the Learning and Teaching of Elementary Subjects

Institute for Research on Teaching

College of Education

Michigan State University

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James W. Reineke

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Abstract

Recent reforms in mathematics education suggest that classroom teachers ought to shift away from the traditional concentration on arithmetical computation and adopt goals of mathematical confidence, problem solving, communication, reasoning, and appreciation. To understand how students are thinking in a given situation, teachers need to make their students' thinking public. One way of accomplishing this is to encourage students to engage in mathematical conversations. In this report, the author describes a classroom where students' thinking was made public through rich discussions in which students presented and justified their interpretations of and solutions to the problems presented in class. In the beginning of this project the author identified the goals the teacher had for her students and then followed the development of those skills and abilities the teacher believed important through a series of classroom observations and teacher and student interviews. The resulting story is, in part, a success story. The teacher and her students learned to talk about mathematics in ways that made their thinking visible and suggested they know mathematics in fresh, inventive ways. But, it also ends with questions about what students learned and what teachers and researchers might consider evidence of their learning.

MAKING CONNECTIONS: TALKING AND LEARNING IN A FOURTH-GRADE MATHEMATICS CLASS

James W. Reineke[1]

In recent years mathematics educators and researchers have called for drastic reform in mathematics education (National Council of Teachers of Mathematics [NCTM], 1989, 1991; National Education Goals Panel, 1992; National Research Council, 1989). These calls have accentuated the belief that mathematics instructors ought to shift away from the traditional concentration on arithmetical computation and adopt goals of mathematical confidence, problem solving, communication, reasoning, and appreciation.

Adopting these goals requires that teachers change their instruction in ways that make their students' thinking public (Putnam, 1992). One way teachers can accomplish this is to encourage their students to engage, among themselves and with the teacher, in mathematical discourse. An emphasis on discourse is evident in both the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) and the Professional Standards for Teaching Mathematics (NCTM, 1991). Across the two documents at least five standards address the importance of discourse in the classroom. These standards suggest communication is a key component in students' construction of mathematical knowledge. The ways children communicate, the reform documents suggest, influence the links students make between their informal knowledge of mathematics and the "abstract language and symbolism of mathematics." (NCTM, 1989, p. 26).

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[1]James W. Reineke, a doctoral student in educational psychology at Michigan State University, is a research assistant in the Center for the Learning and Teaching of Elementary Subjects.

Mathematical discourse is defined in the standards documents as "the ways of representing, thinking, talking, agreeing, and disagreeing" which involves the use of a set of mathematical tools including "special terms, diagrams, graphs, sketches, analogies, and physical models, as well as symbols" (NCTM, 1991, p. 34). What representations are used and how students think and talk, then, influences how students perceive the discipline of mathematics as a domain of inquiry.

In this report, I tell a story about a classroom where students' thinking was made public through rich discussions in which they presented and justified their interpretations of and solutions to the problems presented in class. From September through December 1989, I observed in this classroom twice a week. All of these observations were audiotaped and I wrote extensive fieldnotes of each observation. Some lessons--those either the teacher (Liza)[2] or I thought would be interesting--were videotaped. I interviewed Liza after nearly every lesson. During the interviews I asked about the connections she made among her teaching in this classroom, her previous teaching, and current thinking in mathematics education. We talked at length about what students, in general and some in particular, were learning and what Liza thought constituted evidence of their learning. I also talked with Liza about her goals going into the school year and how they had changed in December. Throughout my 10 weeks in Liza's classroom, I talked informally with students. These conversations focused on their interpretations of the problems on which they were working and the connections they made

[2]Names of teacher and students are pseudonyms.

between in- and out-of-school math. All of these data sources contributed to the story I tell here.

This story, however, is about three lessons I observed from early September through early October. I chose these lessons because they focus on the same content--the composition of functions--and they allow a glimpse of classroom interaction over time, that makes changes in the way students talked about math--and perhaps what they know about functions--visible. Throughout the story I try to make the same connections or use the same themes that guided the conversations I had with Liza; that is, I look at the changes Liza made in her teaching--both from her previous experience and within this classroom--and I look at what students are learning and, drawing from both the curriculum materials and the discipline of mathematics, I look at what might constitute evidence of their learning.

The story is, in part, a success story. Liza and her students learned to talk about mathematics in ways that made their thinking visible and suggested they know mathematics in fresh, inventive ways. It also ends with questions about what students learned and what teachers and researchers might consider evidence of their learning. I begin the story by introducing Liza, the school in which she taught, the curriculum materials she used and her thoughts on teaching for understanding. Following the short introductions I narrate three lessons during which Liza and her students thought and talked about the composition of functions. Finally, I return to the goals Liza had for her students and look at the three lessons with these goals in mind.

The Setting

Liza was a teacher on a mission. After nine years of teaching experience during which she "taught all subjects like your typical elementary school teacher" (Initial interview, 9/7/89), Liza wanted to change the way she taught mathematics. The previous year Liza entered a graduate program in teacher education at Michigan State University. During that year she became interested in the calls for reform in mathematics education. But before she felt comfortable championing the reforms, she wanted to try them out in a classroom. To help her address this concern, she began working with Magdalene Lampert, a professor at Michigan State University who herself has worked to change how mathematics is taught in elementary schools (Lampert, 1985, 1990) and arranged to teach mathematics in a fourth-grade classroom in a nearby school. Liza was responsible for one hour of mathematics instruction four days per week (Monday through Thursday). The collaborating teacher taught the remainder of the curriculum which sometimes included a Friday mathematics class.

The class comprised 23 fourth-grade students--10 girls and 13 boys--with various cultural and ethnic backgrounds. Students from all over the United States and the world attend this school while one or both of their parents pursue degrees at Michigan State University. In this class 12 students were foreign born; the other 11 were born in the United States.

Because of the school's close proximity to Michigan State University, researchers and interested teacher educators often visit. The students, as a result, were used to people coming into their classroom to observe and ask them questions about their school work. This class in particular was used to this kind of attention. The previous year all but two of them were

in a mathematics class taught by Deborah Ball, a teacher educator from Michigan State University who, like Liza, was concerned about teaching mathematics in a way that reflects the calls for reform (Ball, 1989, 1990a, 1990b, in press).

Through her series of readings and meetings with Lampert, Liza began thinking hard about how and what she would teach in this classroom. As an outcome of these meetings, Liza decided to use the Comprehensive School Mathematics Program (CSMP) (CEMREL, 1985), the district's chosen, but not required, curriculum. CSMP is an innovative elementary mathematics curriculum that emphasizes students' understanding of mathematical concepts. A small number of key mathematical strands are revisited often throughout the curriculum with each lesson expected to last one class period.[3] CSMP suggests that each time a concept is revisited it is presented in an expanded form. This pattern holds for the three classes in this story. All three of the lessons focused on the composition of numerical functions. CSMP defines the composition of functions as a series of operations performed on a number and is limited to the operations of addition, subtraction, multiplication, and division.[4]

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[3]For a more detailed analysis of the CSMP curriculum materials, see Remillard, 1990.

[4]CSMP's definition of functions varies slightly from the accepted modern definition of functions. The modern definition holds that functions are the association of exactly one object from one set (the range) with each object from another set (the domain) (James & James, 1976). This definition allows for modern conceptions of continuous functions with discontinuous derivatives. CSMP's definition of functions does not explicitly contain the notions of range and domain. Rather, CSMP defines functions as what might be termed "rational algebraic functions" following Leibeniz's distinction between algebraic and transcendental functions and Euler's distinction between rational and irrational algebraic functions (Kline, 1972). Malik (1980) has suggested that the modern definition may not be appropriate in most classrooms because students have no use or need for continuous functions with discontinuous derivatives. Instead, he suggests using a simpler definition that is consistent with the development of functions. The simpler definition of functions used by CSMP is consistent with the development of functions and, as such, seems like an appropriate definition of functions for this class.

The students were to discover certain characteristics of functions in each of the classes.

Teaching for Understanding

Liza thought her students should leave the class with a fresh understanding of the mathematics they discussed. She was opposed to the traditional math instruction where ideas are taught in discrete units. In its place Liza wanted to help her students seek out and make connections between mathematical ideas. In our first interview she told me she hoped her students would "know mathematics. They'd be able to see connections between mathematical ideas. And even if they didn't see the connections, they would be looking for connections. They would be curious about mathematics" (Initial interview, 9/7/89). Furthermore, Liza hoped to foster in her students a disposition to take part in mathematical conversations. At one point she told me what her ideal student would be:

> A student that wasn't afraid to say what they thought and could articulate what they were thinking. They could answer questions about their thinking. They would be willing to revise their thoughts. They would be able to ask questions of other people. . . . And they would take each others' thinking seriously.

Thus, teaching for understanding for Liza included two seemingly important goals. She hoped her students would be able to make connections between mathematical ideas, and they would develop a disposition to engage in mathematical discussions where they would listen to and use other people's thoughts and ideas.

Liza was also concerned about what she needed to know to get her students thinking about math in these ways. Her concern focused on two things: her knowledge of mathematics and how to set up an environment that promoted the dispositions and skills she wanted. She was unsure how

much her teaching experience would help her in this classroom. During our first conversation she talked about the "hurdles" she would face.

> I think the hurdle of my own knowledge of mathematics--I don't feel like that's something I've been able to do ahead of time, that it's not possible for me to learn everything I need to know about mathematics before I teach this. And that I have to deal with it as it comes up and figure out what it is I don't know and try to learn more about it.

She went on:

> I was thinking about in the classroom environment that I have set up in the past, I really stressed that it was okay for kids to make mistakes. But I realize I never. . . . When kids gave an answer that wasn't right, I never had them explain how they came up with that. The environment was such that, that's wrong, but it's okay to be wrong. It seems that maybe I can use some of those same things that I used in the past to establish an environment where kids felt that it was okay to give an answer, even if they weren't sure if it was right. But to take it one step further now and have kids talk about why they think that that's the right answer. (Initial interview, 9/7/89)

To Liza, teaching for understanding meant fostering in her students a disposition to engage in serious mathematical conversations. Through these conversations she hoped they would make connections between mathematical ideas and understand the "whys" that she thought had been left out of traditional mathematics instruction. She was concerned that her mathematical knowledge and her past experience would not serve her well in this classroom.

The Lessons

In this section I describe three lessons that took place between September 13 and October 4, 1989. Across these lessons there are two paths of change. First, the content becomes increasingly complex over the three lessons. The content of the lessons changes from initial

experiences with simple arrow roads to questioning whether two or more terms and operations can be assigned to a single arrow. As arrow roads--CSMP's representation for numerical functions--are used differently in the three lessons, they also change in the way functions are represented. Functions are commonly represented in three ways (James & James, 1976): a function rule, a set of ordered pairs, and a graph. In the CSMP curriculum, the terms assigned to arrow roads represent the function rule. The beginning and ending numbers that correspond to a certain arrow road are often complied in tables. These tables represent a set of ordered pairs.

If we look at the three lessons Liza taught, functions are represented differently in each of the lessons. The second path of change concerns the interactions among Liza and her students. The first two lessons Liza finds problematic for different reasons. In the first lesson, Liza is concerned with her students' roles in the interaction and in the second she is concerned with her own. In the final lesson, however, Liza and her students have what she accepts as a successful classroom discussion of arrow roads. She and her students talked together about some rather complex mathematical ideas. Looking at these two paths of change illuminates what may have changed over the three lessons and, at the same time, brings up questions about what Liza's students actually learned.

<u>September 13, 1989</u>

Being unsure of her students experience with the CSMP curriculum, Liza reviewed arrow roads at the beginning of this first lesson. The lesson was limited to simple functions--functions with only one term--and functions are represented only as function rules.

On the front chalk board, Liza drew an arrow road.

The students were already familiar with the representation and quickly told about its different parts. One student, Lorenzo, said he had seen things like this in third grade and that they were called "arrow roads." He added that he could do *anything* (any mathematical operation) with arrow roads. Other students agreed and explained the different pieces of the arrow road. Faruq explained the zero saying it was a number on the road from which they could start adding more numbers. Bob told Liza the +2 told them what to do for each arrow. With the arrow road fully explained, the class filled in the empty spaces. On the ends of the arrow roads were two incomplete arrows. Liza asked the class what they represented and the class told her arrow roads go on and on. Liza's question led to a discussion of infinity that she at first encouraged and then quickly shelved.

Liza told the class they could draw points of ellipsis at each end of the road to show that they continue. One student, Pili, suggested that points of ellipsis means the arrow road wouldn't end until it got up "into the thousands or zillions, or something." Pili also suggested that the negative side would have "the same numbers." Liza asked Pili what she meant when she said the negative side would have the same numbers. She and Pili continued to talk until it was clear that the negative side of the arrow road would be filled in with the "same numbers, only negative."

Liza then asked for reactions to what Pili had said. Sipho thought the arrow road would never stop, it would keep on going. Liza asked him why

he thought it would keep on going and Sipho told her the numbers would never stop. He said, "It will go as big as you want." Other students agreed with Sipho, and Liza asked Pili what she thought about Sipho's idea. Pili agreed that the final number moves and added "But it has to stop at a zillion because that's the highest number to go. If you don't stop there, you go back to zero."

Liza again asked the other students for their input. Bob suggested that numbers go beyond a zillion, so they could "go as high as they want." Then Lorenzo announced he had a theory. He claimed,

> There's no number that you can stop at when you're counting. You can always go on. Suppose somebody said, "Guess what the largest number is? One thousand, nine hundred, eighty eight." Well just say, "Add one to that, one thousand, nine hundred, eighty nine and there's always a higher number."

Other students agreed saying there are numbers beyond a zillion and if you run out of numbers you could make up your own. Liza ended the discussion saying "I think this is a question you should continue to think about."

For the rest of this lesson the students looked for "Zap," an imaginary culprit whose identity could be uncovered by filling in arrow roads. CSMP considers this exercise a review of the minicomputer rather than an introduction or review of arrow roads. The lesson, as it is written in the teacher's guide, nas students solve a detective story after collecting clues from arrow roads, the Papy minicomputer, and string pictures.[5] Liza, however, presented all the clues on the arrow road.

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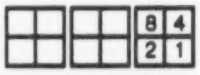
[5]CSMP uses a series of unique representations including minicomputers and string pictures as well as the arrow roads used in these lessons to help students think about the concepts presented. The minicomputer is a tool developed by the Belgian mathematician George Papy. Both binary and decimal in structure, it consists of three or more boards each divided into four sections. Each board represents a "place" in the base ten system and each section a certain denomination. The board farthest to the right represents the ones place with denominations of 1, 2, 4, and 8.

She began this part of the lesson by drawing two more arrow roads--each with its own function rule and one number. Initially, Zap might be any number on either of the arrow roads. Later he became numbers that were found on both arrow roads. To fill in the arrow roads the students needed to follow the rule attached to it. In this class the two arrow roads had rules of +3 and +4. In addition, each of the roads already has a number on it--the +3 road has a 0 and the +4 road has a 1. The students filled in the spaces on each of the roads and decided which numbers could be Zap.

Because her students were in a classroom where mathematical discussion was promoted the previous year, Liza was somewhat intimidated by them. In our conversation after this lesson, she told me:

> I was looking back at what I had written and one of the things that I had written is when Jane [the classroom teacher] had told me . . . that all of the these kids except for two had had Deborah and my response when I wrote about it was that I was nervous and I thought that these kids are going to know more math than I do.

Although it is unclear that her students knew more math than Liza, they may have been more comfortable engaging in mathematical conversations of the sort Liza hoped for.



The middle square would be the tens place equivalent of this square; that is, it would represent 10, 20, 40, and 80. The square furthest to the left would represent the hundreds place. Checkers are placed on the minicomputer to represent a number and "trades" or "moves" can be performed to represent various mathematical operations.

String pictures represent classification. Each string in a string picture represents a set. The intersection of two or more sets are represented by overlapping strings. Above each string is a label that describes the attributes of the set.

One of Liza's goals for her students was to have them develop a disposition for engaging in mathematical conversations. This disposition, according to Liza, had students speaking up in class and articulating their thoughts about the mathematical problems and situations that confronted them. The students would revise their thoughts based on what other people said and did in the conversations. In this lesson Liza's students had a lengthy discussion of infinity. Liza's role in the discussion was to introduce the topic and call on people to speak next. The students, however, engaged in a conversation where they were listening to other students and revising their thinking or supporting their thinking based on things other people said. Consider this short exchange from the larger discussion of infinity:

Pili	I agree that, I agree that it moves, but it has to stop at a zillion because that's the highest number to go. If you don't stop there, you go back to zero.
Liza	Okay, what do other people think about that? Bob?
Bob	I challenge that because you could go up as high as you want, you can go past a zillion.
Liza	And, how do you know that?
Bob	Because there are numbers past a zillion.
Liza	Okay, what do other people think about that? Lorenzo?
Lorenzo	I got a theory.
Liza	What's your theory?
Lorenzo	There's no number that you can stop when you're counting you can always go on. Suppose somebody said "guess what the largest number is? One thousand, nine hundred, eighty-eight." Well just say add one to that, one thousand, nine hundred, eighty-nine, and there's always a higher number.
Liza	Okay, what do other people think about that? Bob?
Bob	I agree because Pili said, um, you can't go past a zillion when you're doing the arrow road, but you can, you can go as high as you can.

Sipho I agree with Bob.

Pili I know that there is a number past a zillion.

Pili, who started this conversation by suggesting that a zillion equals infinity, agrees with someone else that infinity moves. She is not, however, willing to give up all of her beliefs, so, if infinity moves beyond a zillion, it goes back to zero. Bob does not think Pili has changed enough and challenges her statement and Lorenzo offers a theory of infinity. Bob and Sipho agree with Lorenzo and Pili, again, changes the way she talks about infinity. In this short exchange the students are presenting their ideas about infinity, listening to and valuing the ideas of other students, basing what they have to say on what other students have said, and changing their minds when they see fit--all of the things Liza thought her students should do.

Liza didn't say much about the discussion of infinity. She did say she was concerned about the discussion because she had not planned on it. She said "It just seemed to me like the kids were talking, but it wasn't moving anywhere" (Interview, 9/13/89). To end the conversation Liza told the class they should think about infinity on their own. In our conversation Liza questioned whether she would ever come back to this topic. She said:

> When I decided to say that it wasn't because I thought this is really a question I want to pursue later. That's not what made me decide that. I wanted to move out of that topic and that discussion and move on.

Liza's concern over the infinity discussion came from two sources--her knowledge of the lesson's content and how she and her students might go about structuring their interactions. In this lesson Liza believed her

knowledge of arrow roads and her uncertainty about what to expect from her students led to a less than perfect lesson. She told me:

> I feel like I am vague at points. . . . Today I felt vague in . . . expectations for behavior and also, um, the idea of the arrow road. . . . I think I was expecting them to tell me how they've used [arrow roads] in the past. . . . I tried to ask questions to get at how they used [arrow roads]. . . . I was hoping that [the students] would help me. . . . I need to get . . . at the origin of arrow roads and get information about that. (Interview, 9/13/89)

This class started out as a review of CSMP's representation for numerical functions. The class was going to discuss the different components of arrow roads and what each component represented. Because the lesson focused on learning the representation, the functions were limited to those with one term. The discussion, however, erupted into an exciting discussion of infinity--a discussion that bothered Liza. The students carried on the discussion as Liza directed speaking turns. Throughout this discussion Liza's students displayed the dispositions she hoped they would at the end of the year. Liza, however, thought the discussion went nowhere and tabled it to move onto other things. She thought she was "vague" about the behavior she expected from her students and about the content of the lesson. In the next lesson, both Liza and her students continue their search for appropriate ways of interacting in their classroom.

<u>September 20, 1989</u>

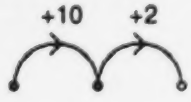
Following the CSMP spiral, the next world of numbers lesson was one week later. In this lesson, the first in a sequence of lessons entitled "The Composition of Functions," students "investigated the composition of certain numerical functions" (CEMREL, 1985, p. 11). CSMP defines the composition of functions as applying a sequence of operations in a given

order. The class was presented with two connected arrow roads, each with a different function rule. By generating a list of numbers that completed the arrow road, students were to discover the pattern that represented the relationship between two sets of numbers--the starting numbers of the first short arrow and the ending numbers on the second short arrow road--and combine the two rules to form one function rule that described the relationship. In this lesson the authors limited the functions to one operation--addition. Functions in this lesson are represented in two ways. They are represented by a function rule that, at first, is two rules and later is combined into one rule, and by a list of ordered pairs in the chart students generated.

Liza drew this arrow road and chart on the front board:

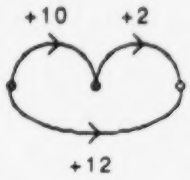
Beginning number	Ending number

Following the script in the CSMP textbook, Liza asked her students to pick a number, add ten and then add two. The class generated a long list of numbers that fit under the beginning and ending circles on the arrow road.



Beginning number	Ending number
99	111
8,000	8,012
250	262
1	13
7	19
203	215
~~4,988~~	5,000
4,988	

When the list was generated, Liza asked the class to look for patterns in the list. Helen told Liza that all the numbers in the second column were 12 more than those in the first. This pattern is the pattern the CSMP teacher's guide says students will discover. Liza agreed and drew in another piece of the arrow road.



At this point, Liza asked the class if they saw other patterns on the chart. David suggested a pattern of 2, 8, 2. Liza did not understand David's pattern and asked him to explain it. David told her you could add two and then eight and then two again to each of the beginning numbers and get the ending number. Liza again told him she did not understand the pattern and asked him to point it out on the board. As David went to the board he said "I'm not sure, but . . ." and pointed out that if you added two and then added eight it was like adding ten. If you added two more you would be doing the same thing as if you were adding ten and then two. Liza said, "Okay" and asked if there were more patterns.

Arif told Liza that about a pattern he saw. He told her that in the first column the eight in 8000, the two in 250, and the two in 203 are also in the ending numbers that correspond to each of those numbers--8012, 262, and 215. Liza acknowledged Arif's pattern by underlining the numbers he mentioned on the chart on the chalkboard. This led to a conversation filled with patterns that might be considered problematic. The patterns students picked out did not apply to the entire list. Rather, they picked out a small group of numbers, that had something in common and presented

that as the pattern. One student suggested that 1, 13, 111, 215, and 19 all had "1s" in them so there was a pattern. These patterns often went down the columns as well as across the table.

In this lesson, Liza writing on the chalk board appeared to validate a student's response. Helen suggested that each ending number is 12 more than its corresponding beginning number and Liza drew in another section of the arrow road. Arif suggested that the 8, 2, and 2 in the beginning column and the 8, 2, and 2 in the ending column constitute a pattern and Liza underlined the numerals in each column. In contrast, David's suggestion that 2, 8, and 2 constituted a pattern met with uncertainty on Liza's part. She repeatedly asked him to clarify the pattern and explain it. When he went to the chalk board to show the pattern, Liza did nothing. What followed these three responses was a series of patterns similar to the pattern Arif pointed out. The students, it seemed, had figured out what was an appropriate response to Liza's question.

But, while her students were trying to parse the lesson, Liza was questioning her participation in the discussion. In our conversation following the lesson Liza told me:

> I felt like I was floundering today and . . . as I stood up there it occurred to me, what do I, asking myself the question, what do I mean by a pattern? What does the word pattern mean? I'm not sure I had the answers to the questions but that's what I was thinking as I stood there listening to these kids. . . . Um, I felt really like I didn't know what to do to move the discussion about patterns and I, that's a part I wasn't pleased about at all, for myself, I felt really uncomfortable with it. . . . Were the examples those kids were giving me, were those patterns? And then it's like, how would I move them, because to me it seemed like, some of the examples they gave me, were looking at the two numbers across from each other. I think what I thought would happen is that they would look at these lists of numbers and look

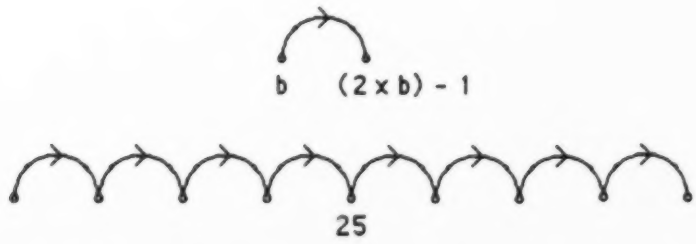
> for patterns in the lists. And then I realized the patterns could go both ways, they could be going down with the numbers, between numbers, starting numbers, and ending numbers. But then it doesn't make sense to me either that there would be patterns going down. But that's what they were, they were giving patterns that way and I didn't know what to do. I didn't, I just didn't know what to do.

This lesson, in contrast to Liza's review of arrow roads and the ensuing discussion of infinity, has both Liza and her students figuring out what counts as an appropriate response. Liza, on one hand, perceives her knowledge of patterns as problematic. As a result she does not know if the things her students bring up in class are patterns and is unsure of how to respond. Her students, on the other hand, worked hard to figure out what counts as an appropriate response during this class. Liza's reactions to different responses gave them clues as to what was acceptable. Once they knew, their responses came much more easily. This is quite different from their engagement in their exciting conversation about infinity in the first lesson. In the final lesson, Liza and her students seem to have constructed ways of communicating in their classroom.

<u>October 4, 1989</u>

The third lesson was a continuation of a lesson that began the day before. On that day Liza gave her class examples of numbers that would fit on an arrow road and they were to guess the function rule. The functions students discussed in this lesson are different in two ways. First, in this lesson, students begin discussing complex functions--those with two or more terms. Second, in contrast to the previous lesson where the functions were limited to one operation (addition), in this lesson two operations are assigned to one arrow and they must be performed in a certain order.

At the start of the third lesson, Liza drew this arrow road on the front chalk board:



The class began discussing a game they played the day before. The game, called "guess my rule," involved guessing what rule allowed them to get from one circle to the next on the arrow road. The students uncovered the rule by investigating an arrow road that had all the numbers provided but no rule or clue as to the origin of the numbers. The students were required to figure out the rule by identifying patterns in the numbers. The rule the students came up with is the rule at the top of today's problem ((2 x b) - 1). The students were instructed to draw a line in their math notebooks, write down today's date, and then fill in the spaces on the arrow road. Bob told her they put the rule down yesterday. Liza told him she would like him to put it down again today by the line he drew in his notebook.

As the students began to work Liza announced that she had placed a number, 25, on the arrow road. She wanted to make sure each student copied the number. The students began to work independently. The room was quiet as Liza walked around and talked with students who had questions. In most cases her first question to the students was "have you talked to someone at your table?"

At one point Liza stopped the class and reminded them of what they had done the day before. They had given her numbers (b) and she had

provided what the next number was ([2 x b] -1). She told them they were doing the same thing today, only that one number was given and they had to figure out the other numbers. It was almost as if the roles had been reversed. She asked the students what they could use for "b." Arif responded that they could use any number for "b." When asked why, he responded with, "You said yesterday that b could be any number." Liza agreed and then pointed to the number she had put on the arrow road and asked how they would use this number. Helen responded that they first needed to find out what the number below it was. Liza asked her how she could figure out that number. Helen responded "You look at the 25 and think 'what numbers could be around 25 on this arrow road?'"

Helen's response led to a large group discussion of the problem. Liza asked what numbers the students had written down that fit on the arrow road. Arif offered 13. When asked where on the arrow road it fit he said just to the left of 25. He explained his choice of that place saying that 13 + 13 = 26 and 26 - 1 = 25, so 13 is the number to the left of 25. Bob challenged, saying that 13 should go one dot further to the left and then 26 and then 25. He supported his solution saying 13 + 13 = 26 on the next dot and then 26 - 1 = 25. His interpretation of the problem, and perhaps arrows roads in general, seemed to be that only one operation or term can be performed per arrow. Mike challenged Bob on this interpretation saying that all the steps are performed for each arrow. Bob wasn't convinced.

David agreed with Bob and explained it by saying "it wouldn't make sense to put the 13 on the dot next to 25 because 13 + 13 = 26 and then you wouldn't subtract the 1." He argued that 13 x 2 = 26 then you subtract the 1 getting 25. If you subtracted the 1 first you would get 12 and 12 x 2 = 24, so the order has to be 13, 26, 25. Mike's argument held to the notion

that one operation corresponds to one arrow. Faruq presented an argument by saying the numbers on the arrow road should be 7, 13, and then 25 because 7 + 7 = 14 and 14 - 1 = 13 and 13 + 13 = 26 and 26 - 1 = 25. At this point, Henrietta wanted to bring something up. Liza attempted to direct the conversation here by asking whether the idea Henrietta was bringing up was about what Faruq had said. Henrietta said "Well, sort of. I still don't get it." She said that the rule had 2 x b in parentheses and then you subtract 1. Liza left it here and asked the class if they could return to Faruq's comments.

Liza used Faruq's comments to review what they had done during "Guess my rule" the day before. She told them the first couple of problems had only one step (operation) per arrow. But the last game had two steps per arrow. She used this example to show that two steps can correspond to one arrow. While she pointed to Bob's solution, Liza asked the students about how many steps there are in the solutions to the problem. The students answered "Two." "How many arrows?" Liza asked. After a long pause, the students answered "Two." Liza then asked if that was the same as what they had done yesterday. Mike told her they used two steps for one arrow the day before and today they were using two arrows for the two steps. Liza said "YES! We did." The problem was clarified by Liza and her students; they agreed that two steps could be applied to one arrow.

When the class had agreed on the interpretation of the problem, Liza asked what the number to the left of 25 should be. David told her by the accepted interpretation it would be 13, but if you did it the way we thought you wanted us to (Bob's interpretation), it would be 14.

Henrietta extended the arrow road to the left. She said the left side should be filled in with the numbers $2\frac{1}{2}$, 4, 7, 13, 25. Asim agreed saying he got the same thing. Liza asked him how he came up with $2\frac{1}{2}$. Asim told her he got stuck at 4 but then he remembered halves. He said he tried all the halves and $2\frac{1}{2}$ worked. "Two plus 2 is 4 and one half...um, two halves is 5 and 5 - 1 = 4." WuLee challenged Asim's explanation saying that $2\frac{1}{2} + 2\frac{1}{2} = 4\frac{1}{2}$. Liza asked her how she had arrived at that answer and then walked her through the series of additions she had done. At the end WuLee realized her mistake and revised her thinking--$2\frac{1}{2} + 2\frac{1}{2}$ was 5, not $4\frac{1}{2}$. This discussion continued for a short time with the students presenting possible solutions and challenging each other's ideas. Throughout this discussion, Liza asked students to clarify their responses and tell how their comments were connected to previous comments. Once the problem was clarified the students again began to work in their notebooks. Liza moved from student to student and asked and answered questions. Most of her questions were directed at having the students explain their responses or procedures.

Earlier in the year Liza mentioned she was concerned about setting up an environment where students talked about mathematics. In this class the students are doing just that. The discussion about the numbers on this arrow road was among Bob, Mike, WuLee, Henrietta, David and other students. Liza's role was to keep the discussion focused on the problem. To focus the discussion Liza asked students if their comments addressed issues being discussed. At times she summarized parts of the conversation to help students remember what issues were "on the table"

or to bring something back into the conversation that might help them reach consensus on the topic. When she did not understand a student's explanation of the problem or a solution, Liza often asked probing questions to better understand what the student was saying. Further, she felt freer to tell students if their response was good or right--when Arif suggested they were doing something different then they were the day before, Liza told him he was right. When WuLee had difficulty adding $2\frac{1}{2} + 2\frac{1}{2}$, Liza walked her through the problem until WuLee saw her error.

In our conversation following this class Liza told me:

> I feel like I started this out being very naive thinking that O.K., I am going to follow CSMP and then I was very disappointed when the class did not go the way that CSMP predicted and you know that whole business. So today as I was thinking about going into this class like we talked before and I saw you in the hall before the class and like, oh, my god, I am going to bring up patterns and there I knew that I was going to bring up patterns and I knew that it was going to be an open discussion. I didn't know what was going to happen but, and that makes me anxious not knowing what is going to happen but at least I recognize that now and I feel like I am sort of prepared for that. . . . So I felt like that also was going to be, there was going to be stuff there that I wasn't really sure about . . . maybe what it is, is that I recognize the uncertainty. I don't know.

Summary and Discussion

The conversations Liza and her students had changed across these three lessons. The change culminates in the last lesson where they have an exciting discussion about complex functions using two operations and how they can be represented by arrow roads. In that discussion, Liza's students display the dispositions and skills she hoped her class would display at the end of the year--they eagerly and skillfully engage in mathematical conversations. But does their participation in this

discussion provide evidence for their learning? After all, Liza's students were engaged in an exciting mathematical conversation during the first lesson.

One interpretation of this change is that over the course of these three lessons the class developed a set of interactional norms that allowed them to engage in these conversations. Edwards and Mercer (1987) have suggested that classroom conversation is "an instance of talk in general" (p. 42). As such, classroom interaction is framed by local versions or instantiations of the co-operative principle (Grice, 1975). This principle holds that people involved in a conversation will (a) contribute only what they have evidence for and believe to be true, (b) provide only the amount of information that is necessary, (c) make their contribution relevant to the conversation, and (d) make their contribution intelligible. What each of these maxims actually mean in practice is dependent on the particular social situation in which they are used; that is, what these maxims look like in a given classroom emerges through participation in classroom discourse.

The development of a cooperative principle takes time. Although most members of this class were in Deborah's Ball's class the year before and had developed ways of interacting that included the dispositions Liza valued, together they needed to figure out appropriate ways of interacting. Over the three lessons presented here, Liza and her students developed ways of interacting that allowed them to engage together in mathematical discourse. Liza came to understand better what her role ought to be in this class, and her students came to understand what would be accepted as an appropriate response. Although her students were able to engage in a mathematical discussion in the first lesson, Liza felt uncertain about

her role and abruptly stopped the conversation with no intention of returning to the topic of infinity. In the second lesson, Liza and her students were still looking for appropriate ways to interact. But, in the third lesson, Liza and her class seemed to agree on the norms of interaction for the class.

Although Liza and her students were engaged in exciting conversations about arrow roads and the number of operations that can be assigned to each arrow, these lessons give rise to questions about what it might mean to engage in mathematical conversations. Liza's second goal for her students was for them to make connections among mathematical ideas. Do the conversations Liza had with her students provide evidence of these connections?

CSMP's authors suggest that mathematics for young children can, and should, be taught using three languages--the language of strings, the language of arrows, and the Papy minicomputer. These languages, they suggest, allow students to learn mathematics without being encumbered by the abstract "natural" languages used in the discipline of mathematics. By talking about arrow roads, they suggest, students will construct important ideas that can later be labeled as functions. As a result, Liza and her students never use the term "function" in their conversations. Instead, they used language that allowed them to manipulate CSMP's representation of the relationship between two or more numbers without concern for the fit with the discipline of mathematics.

In contrast, some researchers and educators suggest that learning a discipline includes learning the ways of interacting that are unique to that discipline. Jay Lemke (1990), for instance, begins the first chapter of his book Talking Science this way:

> Learning science means learning to *talk* science. It also means learning to use the specialized language in reading and writing, in reasoning and problem solving, and in guiding practical action in the laboratory and in daily life. It means learning to communicate in the language of science and act as a member of the community of people who do so. (p. 1)

Can we say the same thing about mathematics? Do students need to "talk mathematics" in ways that reflect the discourse of mathematicians? David Pimm (1987) in his book Speaking Mathematically, explains: "Part of learning mathematics is learning to speak like a mathematician, that is, acquiring control over the mathematics register." (p. 76). The mathematical register, according to Pimm, is a collection of terms and ways of speaking commonly used in the discipline of mathematics. Many words have everyday meanings that are different from the meanings assigned to those words when they are used in the discipline of mathematics (Pimm, 1987; Walkerdine, 1988). Words like "more," "less," "right," odd," and even "function" have very different meanings in everyday interactions and mathematical interactions. To learn mathematics, Pimm argues, means learning the mathematical meanings of these terms and using them in ways that are appropriate in the discipline.

Placing these two conceptions of mathematics side by side raises some questions. If we accept the notion that mathematical terms need to be used in mathematical conversations, can the conversations Liza and her students had be considered mathematical? After all, they did not use the term "functions" in their conversations. They did, however, express the relationship between two sets of numbers as a number sentence, compile a list of ordered pairs and talk about the number of operations that can

represent the relationship between two sets. And, they did this through a series of conversations where they explored CSMP's representations, made conjectures, challenged each others thinking, and converged on an interpretation of a mathematical problem, situation, or solution. These things are compatible with a conception of doing mathematics that Lakatos has called "quasi-empirical investigation" (Lakatos, 1986). Quasi-empirical investigation according to Lakatos, "Starts with problems followed by daring solutions, then by severe tests, refutations. The vehicle of progress is bold speculations, criticism, controversy between rival theories, problem shifts" (pp. 34-35). Liza and her students were involved in conversations that include all of these components. Mathematical discourse in quasi-empirical investigation does not appear to be limited to or even need to include the terminology of mathematics (i.e., the mathematics register). Rather, what constitutes mathematical discourse is a process, a process that consists of ways of investigation, development of academic argument, and serious refutation by other people.

Based on those things, can we say Liza's students attained the goal of making connections among mathematical ideas? If mathematical discourse is defined as knowing and using a certain mathematical register, then perhaps Liza's class did not learn about functions. But if mathematical discourse is characterized by quasi-empirical investigation, then perhaps exploring functions without using the mathematical register can still result in students becoming more knowledgeable about functions.

Conclusions

The story of Liza and her students is in a sense a success story. Over the course of these three lessons they learn how to engage in exciting

conversations about arrow roads. Rather than attributing that skill to either improvement in Liza's teaching or her students' ability to engage in mathematical conversations, it seems they constructed norms of interaction together that allowed and even encouraged them to engage in these conversations. At first glimpse, Liza and her students construct knowledge of functions and they progress through more complex information across the three lessons. Looking more closely, however, illuminates the fact that neither Liza nor her students use the term "functions" anywhere in the lessons. This fact brings up questions about whether students need to know and use the terminology common in the discipline of mathematics. What constitutes mathematical discourse, however, is not clear. Some conceptions of it require that participants in mathematical discourse must share a common language. Others suggest participants need to take part in a process that characterizes investigations in the discipline of mathematics.

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